Freshman Meet 2 – December 11, 2013
Round 1: Algebraic Word Problems

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Suppose you have the same number of nickels as you have quarters. The total value is \$3.30. Find the number of nickels.

2. Today, Emily is four years older than 1/9 her mother's age. In six years, she will be 1/3 her mother's age at that time. How old, in years, is Emily today?

3. A reseller spent \$100 for several copies of the same book. If the price had been \$2.50 less per book, he could have bought 2 more for the \$100. How much did he actually pay per book?

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________ years

(3 pts.) 3. \$ ______________________

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Round 2: Number Theory

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

Note: a subscript indicates that number's base.

1. Express 236_7 in base 10.

2. Find the smallest integer greater than 500 that is divisible by 2, 3, 5, and 9, but is not divisible by 4.

3. If $N = 3 \times 6 \times 9 \times 12 \times 15 \times \ldots \times 42$ is divisible by 6^k, find the largest possible integer value of k.

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

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Round 3: Operations on Numerical Fractions, Decimals, Percents, and Percentage Word Problems

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If 80% of the class was marked excellent in spelling and 60% of these excellent pupils received honor certificates, what percentage of the whole class received honor certificates?

2. Express $35\frac{5}{7}\%$ as a fraction in simplest form.

3. Evaluate and express as a simplified fraction:

$$\left[\frac{2\frac{1}{2}\times 3}{1\frac{3}{5}\div 3.1\overline{9}}\right]\times\left[2.8\overline{3}-\left(\frac{2}{9}+1\frac{5}{18}\right)\right]$$

The lines above the digits denote repeating decimals.

ANSWERS

(1 pt.) 1. ______________________ %

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

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Round 4: Set Theory

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Given $P = \{2, 3, 4, 5\}$ and $Q = \{1, 3, 4, 6\}$, find the number of proper subsets of $P \cap Q$.
(A subset which is not equal to the original set is called *proper*.)

2. The universal set for this problem is $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $A = \{0, 2, 4, 6\}$, $B = \{1, 2, 3, 4, 7\}$, and $C = \{6, 7, 8, 9\}$. If S' denotes the complement of S, find the set

$$(A' \cap C) \cap (B \cup C').$$

3. A truck contains 70 rugs, all of which are made of wool or cotton and are blue or brown in color. If 26 are blue wool rugs, 40 are cotton rugs, and 13 are brown rugs, how many are brown cotton rugs?

ANSWERS

(1 pt.) 1. ________________

(2 pts.) 2. { ________________ }

(3 pts.) 3. ________________ brown cotton rugs

Freshman Meet 2 – December 11, 2013
TEAM ROUND

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (3 POINTS EACH)

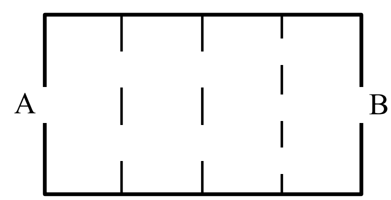
APPROVED CALCULATORS ALLOWED

1. Suppose $A \cap B$ has 8 subsets and $A \cup B$ has 32 subsets. If sets A and B each have the same number of subsets, how many subsets does B have?

2. Twenty mathletes stand in a long column, one behind the other. At a signal from the coach, the mathletes standing in positions 20 and 10 step forward into places 1 and 2, respectively, and the others step back. Find the least number of these maneuvers that must occur before the mathlete originally in front will be in front once again.

3. If $n!$ denotes n factorial, reduce $\dfrac{100!}{102! - 101!}$ to the simplest possible fractional form.

4. How many distinct paths can be walked through this house entering at door A and exiting at B without passing through any interior doorway more than once?



5. Given the following:

   30% of math team questions are strange and easy
   50% of questions are strange but not easy
   10% of questions are not strange but are easy
   10% of questions are neither strange nor easy

   What percentage of strange math team questions are easy?

6. When a rectangular tablecloth was hung over a square table, 3 inches of cloth hung over each of two opposite sides and 2 inches hung over the other two sides. If the area of the tablecloth was 2024 in^2, find the side length of the table, in inches.

7. If y is a real number and $10^{2y} = 25$, find the value of 10^{-y}.

8. The cube of a positive integer n has 4 digits, the last of which is a 3. Find n.

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TEAM ROUND ANSWER SHEET

1. ____________________

2. ____________________

3. ____________________

4. ____________________

5. ____________________ %

6. ____________________ inches

7. ____________________

8. ____________________

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ANSWERS

ROUND 1
(Bromfield, West Boylston, Bancroft)

1. 11
2. 8
3. \$12.50

ROUND 2
(Southbridge, Douglas, Worc Acad)

1. 125
2. 630
3. 11

ROUND 3
(Tahanto, Tantasqua, Tantasqua)

1. 48%
2. 5/14 (only)
3. 20

ROUND 4
(St. John's, Quaboag, Quaboag)

1. 3
2. {7}
3. 9

TEAM ROUND

(St. John's, Westborough, [unknown school], Worc Acad, St. John's, Bancroft, Notre Dame, St. John's)

1. 16
2. 15
3. 1/10201 (only)
4. 36
5. $37.5\% = 37\frac{1}{2}\%$
6. 40 inches
7. $1/5 = 0.2$
8. 17

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FULL SOLUTIONS

ROUND 1

1. Combined, a nickel and a quarter are worth 30¢. In \$3.30, there are $330/30 = 11$ such sets. Therefore, you have $\boxed{11}$ nickels.

2. Let Emily's current age be x years. Express her mother's current age in terms of Emily's age in two different ways, and then solve for x.

$$\begin{aligned} 9(x-4) &= 3(x+6)-6 \\ 9x-36 &= 3x+12 \\ 6x &= 48 \\ x &= \boxed{8}. \end{aligned}$$

3. Let the actual price per book be x dollars. Then,

$$\begin{aligned} \frac{100}{x}+2 &= \frac{100}{x-2.5} \\ 100x &= (100+2x)(x-2.5) \\ 100x &= 100x-250+2x^2-5x \\ 0 &= 2x^2-5x-250 \\ 0 &= (2x-25)(x+10) \end{aligned}$$

   Therefore, $x = \boxed{\$12.50}$.

ROUND 2

1. We have that $236_7 = 2(7^2)+3(7)+6 = \boxed{125}$.

2. If a number is divisible by 2, 3, 5, and 9, it is also divisible by their lcm, which is 90. The smallest integer multiple of 90 that is greater than 500 is 540, but 540 is divisible by 4. The next smallest, $\boxed{630}$, is not.

3. Factoring out a factor of 3 from each term, $N = 3^{14}\cdot 14!$. We wish to find the largest k such that 6^k divides $3^{14}\cdot 14!$. Since $6 = 2\cdot 3$, we need to find the number of factors of 2 and 3 in N and take the lesser value.

   We know that 3^{14} contains no factors of 2, while 14! contains $7+3+1 = 11$ factors of 2. This is less than the 14 factors of 3 in 3^{14} alone, so $\boxed{11}$ is our answer.

ROUND 3

1. Convert to decimals and multiply: $0.8 \cdot 0.6 = 0.48 \equiv \boxed{48\%}$.
2. First, convert the mixed number into an improper fraction: $35\frac{5}{7} = 250/7$. Since this is a percentage, divide by 100 to find that $35\frac{5}{7}\% = \dfrac{250}{700} = \boxed{\dfrac{5}{14}}$.
3. First, convert all of the repeating decimals and mixed numbers to (improper) fractions[1]:

$$\left[\frac{2\frac{1}{2} \times 3}{1\frac{3}{5} \div 3.1\overline{9}}\right] \times \left[2.8\overline{3} - \left(\frac{2}{9} + 1\frac{5}{18}\right)\right] = \left[\frac{\frac{5}{2} \times 3}{\frac{8}{5} \div \frac{16}{5}}\right] \times \left[\frac{17}{6} - \left(\frac{2}{9} + \frac{23}{18}\right)\right].$$

For the first bracket, multiply all of the fractions together, inverting as necessary for division. Most of the terms cancel.

$$\left[\frac{\frac{5}{2} \times 3}{\frac{8}{5} \div \frac{16}{5}}\right] = \frac{5}{2} \cdot 3 \cdot \frac{5}{8} \cdot \frac{16}{5} = 15.$$

Perform the necessary addition and subtraction in the second bracket:

$$\left[\frac{17}{6} - \left(\frac{2}{9} + \frac{23}{18}\right)\right] = \frac{17}{6} - \frac{27}{18} = \frac{8}{6} = \frac{4}{3}.$$

Finally, multiply the two: $15 \cdot \dfrac{4}{3} = \boxed{20}$.

ROUND 4

1. First find that $P \cap Q = \{3, 4\}$ contains 2 elements. Therefore, there are $2^2 = 4$ subsets, of which $4 - 1 = \boxed{3}$ are proper.

   (*Proper* subsets of a set include the empty set, but not the set itself.)
2. Evaluate the expressions in the parentheses first. $A' \cap C = \{7, 8, 9\}$ and $B \cup C' = \{0, 1, 2, 3, 4, 5, 7\}$. The intersection of these two sets is $\boxed{\{7\}}$.

[1] Here, a simple shortcut is to realize that $3.1\overline{9} = 3.2$. In general, the strategy for problems with repeating decimals is to subtract such that the repeating portions cancel out. In this case, let $x = 3.1\overline{9}$. Then, $10x = 31.\overline{9} = 31.9\overline{9}$. Subtracting, $9x = 28.8$ so $x = 3.2$.

3. Make a table with the given information:

	wool	cotton	
blue	26		
brown			13
		40	

The total number of rugs is given to be 70, so $70 - 26 = 44$ rugs are cotton or brown. There are 40 cotton rugs and 13 brown rugs, so by inclusion-exclusion, $40 + 13 - 44 = \boxed{9}$ must be both brown and cotton.

TEAM ROUND

1. A set of n elements contains 2^n subsets. Therefore, $A \cap B$ contains 3 elements and $A \cup B$ contains 5 elements. Since A and B contain the same number of elements, each must contain 4 elements. Therefore, B has $2^4 = \boxed{16}$ subsets.

2. Keep track of the position of the mathlete who started in the 1^{st} position. When she is in the 1^{st} through 9^{th} positions, each maneuver causes her to move backwards by 2 positions. When she is in the 11^{th} through 19^{th} positions, she moves backward by 1 position (the person in position 10 was already in front of her). Therefore, the first mathlete moves as follows:

   $1 \to 3 \to 5 \to 7 \to 9 \to 11 \to 12 \to 13 \to 14 \to 15 \to 16 \to 17 \to 18 \to 19 \to 20 \to 1.$

   Counting, $\boxed{15}$ maneuvers were needed.

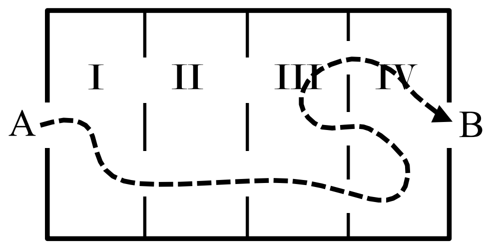
   [Side-Note: With 20 mathletes conducting the same maneuver, it would take at most 20 maneuvers to return the 1^{st} mathlete to the front. To see why, consider the implications if it required > 20 maneuvers. Label each mathlete 1–20 based on his initial position in the line. After each maneuver, keep track of the number of the mathlete currently in the 1^{st} position. This cycle must have length 20 at maximum. If it had length > 20, then by the Pigeonhole Principle, some mathlete must have appeared twice in that cycle. But then that mathlete would define a shorter cycle. Since there is nothing special about any of the mathletes, this contradiction completes the proof.

   It is possible, however, that it would require fewer than 20 maneuvers. In this case, the cycle has length < 20 and some of the mathletes never make it to the front. In this problem, note that the five mathletes in positions 2, 4, 6, 8, and 10 are in their own cycle and never appear in the 1^{st} position.]

3. Factor the expression:

$$\begin{aligned}\frac{100!}{102! - 101!} &= \frac{100!}{(102-1)\cdot 101!}\\ &= \frac{1}{101\cdot 101}\\ &= \boxed{\frac{1}{10201}}.\end{aligned}$$

4. The "walls" split the house into four regions, as shown in the figure below.



The wall separating regions I and II contains two doors. A path from door A to door B must choose one of these doors and then proceed to the next region; since only two doors are available, the path would be stuck in region I if both doors were used. The same applies for the wall between regions II and III. However, the wall between regions III and IV contains 3 doors. One possibility is that one of the three doors is chosen and then the path immediately exits through door B. Another possibility, using all three doors on that wall, is illustrated above. This strategy also yields a third path in which the initial return and final exit into region IV are reversed. Therefore, there are 3 paths from region III to region IV using the bottom door as the initial door; the other two doors can also serve as the initial door for six more paths.

There are 2 ways to pass from region I to II, 2 ways from region II to III, and 9 ways from region III to IV, for a total of $2 \cdot 2 \cdot 9 = \boxed{36}$ possible paths.

5. From the given information, $30\% + 50\% = 80\%$ of the math team questions are strange, and 30% of the math team questions are strange <u>and</u> easy. Therefore, $\frac{30\%}{30\% + 50\%} = \boxed{37.5\%}$ of strange questions are also easy.

6. Let the side length of the table be x inches. Then, the tablecloth has dimensions $(x+6)\times(x+4)$. We are given that this is equal to 2024.

$$\begin{aligned} (x+6)(x+4) &= 2024 \\ x^2+10x+24 &= 2024 \\ x^2+10x-2000 &= 0 \\ (x-40)(x+50) &= 0 \\ x &= \boxed{40}. \end{aligned}$$

7. **METHOD I:** If $10^{2y}=25$, then $10^y=\sqrt{25}=5$. Therefore, $10^{-y}=5^{-1}=\boxed{1/5}$. Note that taking the negative square root and setting $10^y=-5$ gives no real solutions in y.

   METHOD II: Taking the logarithm of both sides, $2y=\log 25=2\log 5$. Therefore, $y=\log 5$ and $10^{-y}=\boxed{1/5}$.

8. If n^3 has 4 digits, then $\sqrt[3]{1000}\le n<\sqrt[3]{10000}$, implying that $10\le n<21.5$. Also:

last digit of n	last digit of n^3
0	0
1	1
2	8
3	7
4	4
5	5
6	6
7	3
8	2
9	9

   Therefore, since we are given that n^3 ends in a 3, $n=\boxed{17}$.